6/11/17

High dim expanders and group stability - Lubotzky

Property (T)

Let Γ be a countable discrete group. Γ has (T) if $\exists$ a finite subset $S \subseteq \Gamma$ and $\varepsilon = \varepsilon(\Gamma, S) > 0$, s.t. $\forall$ unitary rep (H, ρ) $\rho: \Gamma \to \mathcal{U}(H)$ if $\exists v \in H$ $\|v\| = 1$ s.t. $\|\rho(s)v - v\| < \varepsilon$ $\forall s \in S$, then $\exists v' \neq 0$ s.t. $\rho(g)v' = v'$ $\forall g \in \Gamma$.

Kazhdan proved that all lattices (discrete subgp. of finite volume) in simple Lie group of rank ≥ 2 satisfy property (T).

<u>Ex</u>: $\Gamma = SL_d(\mathbb{Z})$ $d \geq 3$.

<u>Prop</u>: If Γ has (T) then Γ is finitely generated

<u>Proof</u>: Look at the representation of Γ on $L^2(\Gamma/\Lambda)$, where $\Lambda = \langle S \rangle$. Take $\mathbb{1}_{e\Lambda}$, and observe that

$\forall s \in S \subset \Lambda$ $\quad \rho(s)\mathbb{1}_{e\Lambda} - \mathbb{1}_{e\Lambda} = 0 \Rightarrow$ a fixed point

$\Rightarrow \mathbb{1}_{e\Lambda}$ is a fixed point of $S \Rightarrow \exists f \in L^2(\Gamma/\Lambda)$ which is Γ-invariant. f must be the constant function on Γ/Λ and therefore Γ/Λ must be finite. (otherwise ther const. func. is not in $L^2(\Gamma/\Lambda)$)

$\Rightarrow \Gamma$ is finitely generated. ▨

Thm (Margulis '73): Assume $\Gamma = \langle S \rangle$ has property (T) w.r.t. S satisfying $S = S^{-1}$ and let $\mathcal{L} = \{ N \lhd \Gamma \mid [\Gamma : N] < \infty \}$. Then $\{ Cay(\Gamma/N; S) \}_{N \in \mathcal{L}}$ is a family of ε'-expanders which are $k = |S|$-regular.

Proof: Let $X = Cay(\Gamma/N; S)$ and let $Y \subset X$. Observe the vector

~~$\forall_Y \in L^2(\Gamma/N)$, Γ acts on $L^2(\Gamma/N)$~~

$$f_Y(x) = \begin{cases} |Y^c| & x \in Y \\ -|Y| & x \in Y^c \end{cases} \in L^2_0(\Gamma/N) \rightarrow \sum_x f_Y(x) = 0$$

Γ acts on $L^2_0(\Gamma/N)$. If $\partial Y = \{ \{x,y\} : \begin{matrix} x \in Y, y \notin Y \\ \text{or} \\ x \notin Y, y \in Y \end{matrix} \}$ is small

$\frac{f_Y}{\|f_Y\|_2}$ is an almost invariant $\underset{(T)}{\Rightarrow}$ there is an invariant vector $V \neq 0$ in $L^2_0(\Gamma/N)$. This is a contradiction since this vector must be constant and $\perp \mathbb{1}$. ▨

Group cohomology

Let Γ be any group and V a Γ-module (i.e. V is an Abelian gp with homomorphism $\rho : \Gamma \to Aut(V)$)

Chain complex $\forall n \geq 0$, define $C^n(\Gamma; V) = \{f : \Gamma^n \to V\}$. In particular $C^0(\Gamma, V) = \{V\}$ as $\Gamma^0 = \{*\}$. Coboundary $d \equiv d^n : C^n \to C^{n+1}$ is given by

$$d^n f(g_1, \dots, g_{n+1}) = \rho(g_1) f(g_2, \dots, g_{n+1}) + \sum_{j=1}^{n-1} (-1)^j f(g_1, \dots, g_{j-1}, g_j g_{j+1}, g_{j+2}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n)$$

Prop: $d^{n+1} \circ d^n = 0$ (Exercise)

Defn: $B^n(\Gamma; V) = im\, d^{n-1}$ $Z^n(\Gamma; V) = ker\, d^n$.

By the proposition $B^n \subset Z^n \Rightarrow$ We can define the cohomology group $H^n(\Gamma; V) = Z^n(\Gamma; V) / B^n(\Gamma; V)$.

Look at $d^0 : C^0 \to C^1$.

For $f \in C^0$, $f : * \to v \in V$

$$d^0 f(g) = \rho(g) v - f(*) = \rho(g) v - v$$

Properties: If Γ acts on V without a fixed point, then $\ker d^0 f = 0$.

Indeed, if $d^0 f = 0 \Rightarrow \rho(g)v - v = 0 \quad \forall g \in \Gamma \Rightarrow v$ is a fixed point of the Γ action on V.

More generally $Z^0 = \ker d^0 = \{\text{Fixed points}\}$.

$$B^1 = \{f : \Gamma \to V : \exists v_0 \in V \text{ so that } f(g) = \rho(g)v_0 - v_0\}$$

$$Z^1 = \{f : \Gamma \to V : d^1 f(g_1, g_2) = 0 \quad \forall g_1, g_2 \in \Gamma\}$$

Since $\quad d^1 f(g_1, g_2) = \rho(g_1) f(g_2) - f(g_1 g_2) + f(g_1) \quad \forall g_1, g_2 \in \Gamma$

$$Z^1 = \{f : \Gamma \to V : f(g_1 g_2) = f(g_1) + \rho(g_1) f(g_2) \quad \forall g_1, g_2 \in \Gamma\}$$

<u>Semi direct product</u> $\Gamma \curvearrowright V$

$G = V \rtimes \Gamma = \{(v, g) : v \in V, g \in \Gamma\}$ with the group operation

$$(v_1, g_1) \cdot (v_2, g_2) = \left[v_1 g_1 v_2 g_2 = v_1 g_1 v_2 g_1^{-1} g_1 g_2 = v_1 v_2^{g_1} g_1 g_2 \right] = (v_1 + \rho(g_1) v_2, g_1 g_2)$$

Properties: $V, \Gamma \leq G$, $V \cap \Gamma = \{e\}$, $V \triangleleft G$, $V \cdot \Gamma = G$ and the inner action of Γ on V inside G is the original action.

$V \rtimes \Gamma \xrightarrow{\pi} \Gamma$. Assume $\varphi : \Gamma \to V \rtimes \Gamma$ is a section,
$(v,g) \longmapsto g$

namely a homomorphism s.t. $\pi \circ \varphi = id_\Gamma$ (e.g. $\varphi_0(g) = (e_V, g)$
while in general $\varphi(g) = (\sigma(g), g)$ for some ~~function~~ homomorphism $\sigma : \Gamma \to V$.
satisfying certain condition.

~~Claim: $\varphi : \Gamma \to V \rtimes \Gamma$ is a section iff σ is 1-cocycle.~~

Claim: A map $\varphi : \Gamma \to V \rtimes \Gamma$ $\varphi(g) = (\sigma(g), g)$ is a section (i.e. homomorphism)
iff σ is 1-cocycle.

Proof: (Exercise)

When is such a section a coboundary?

Claim: σ is a coboundary iff φ is conjugate in $G = V \rtimes \Gamma$
to φ_0 ($\varphi_0(g) = (0_V, g)$!). by an element of V

Proof: Check first that conjugating φ_0 by $(-v_0, e_\Gamma)$ and show that
$\tilde{\varphi}(g) = (\rho(g)v_0 - v_0, g)$.

<u>Thm:</u> Γ has (T) iff $H^1(\Gamma;V)=0$ $\forall$ unitary representation $\Gamma \to \mathcal{U}(V)$ (V a Hilbert space).