OPEN QUESTIONS SESSION

SDE-BOKER 2017

1. Nati Linial

1.1. Finite Projcetive Planes. A finite (combinatorial) projective plane is a set of points and lines (P, L) with an incidence function $\mathrm{Inc}\colon P \times L \to \{T, F\}$, where every two points $p_1, p_2 \in P$ determine a unique line $l(p_1, p_2) \in L$ and every two lines $l_1, l_2 \in L$ determine a unique point $p(l_1, l_2) \in P$.

For every finite field $\mathbb{F}_q$, q a prime power, the 1-dim and 2-dim linear subspaces of $\mathbb{F}_q^3$ interpreted as points and lines define a projective plane with $q^2 + q + 1$ lines (and points). There are known constructions that are different from the above. However, the order of every known projective plane is, as above of the form $q^2 + q + 1$. A well-known open problem asks:

Question 1.1. *Can we find combinatorial projective planes in which the number of lines (points) is not of the form $q^2 + q + 1$ for q a prime power?*

It is known (through very extensive computer search) that there exist no projective plane of order $q = 6$ or 10. There are also additional arithmetic restrictions known, but the general question is open.

Do recent advances in the study of combinatorial designs give us new ideas on how to attack this problem?

1.2. Discrepancy (see J. Matoušek's beautiful book [Mat99]**).**

(1) If you are asked to place n points uniformly in $[0, 1]$ - That is easy.
(2) We move on to $[0, 1]^2$. If $Z \subset [0, 1]^2$ is an n-point set, we define its *discrepancy* as follows:

$$\mathrm{disc}(Z) = \max \left| |R \cap Z| - n \cdot \mathrm{area}(R) \right|$$

where the maximum is over all rectangles R in $[0, 1]^2$. It is known that the least discrepancy of an n-point set is $\Theta(\log n)$.
(3) You can formulate the same kind of question for n-point sets in the 3-cube, where R is now a box. (Conceivably the least distortion for a 3-dimensional n-point set is $\log^2 n$).

1.3. **The Signing Conjecture - Bilu, Linial.** Let $G = (V, E)$ be a d-regular graph and let A be its adjacency matrix. A *signing* of G is a symmetric matrix that is attained by replacing some of the 1 entries in A by -1.

Conjecture 1.1. *Every d-regular graph has a signing all of whose eigenvalues are in* $[-2\sqrt{d-1}, 2\sqrt{d-1}]$.

Proof of this conjecture would yield the existence of d-regular Ramanujan graphs for all $d \geq 3$. The context of this problem is that of 2-covers of graphs.

Remark 1.2. *Marcus-Spielman-Srivastava proved this for bipartite grpahs. The non bipartite case is still open.*

1.4. **Sparsity of Simplicial Complexes.**

Question 1.2. *How sparse can a simplicial complex be?*

Specifically, find the largest $f = f(n, t, k)$ *such that every* n*-vertex, two-dimensional complex with* $t = \#2$*-dim faces must have a set of* k *vertices that spans at least* f *2-dim faces. Are there* n*-vertex* t*-face complexes that are at once optimal for a wide range of* k*'s?*

2. Amitay Kamber

2.1. **Ramanujan Surfaces.** Let $\mathbb{H}$ be the hyperbolic plane. $\Gamma_0 = \mathrm{PSL}_2(\mathbb{Z})$ acts nicely on it. A congruence subgroup of Γ_0 is a subgroup of the form

$$\Gamma_0(N) = \ker \varphi_N \colon \mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z} \pmod N).$$

For any finite index $\Gamma \leq \Gamma_0$, consider the quotient w.r.t the action on $\mathbb{H}$, $X_\Gamma = \mathbb{H}/\Gamma$ and consider its Laplacian $\Delta \colon L^2(X_\Gamma) \to L^2(X_\Gamma)$. We know that $\mathrm{Spec}_\Delta(\mathbb{H}) \subseteq [\frac{1}{4}, \infty)$.

We say that X_Γ is Ramanujan if

$$Spec_\Delta(X_\Gamma) \subseteq \{0\} \cup [\frac{1}{4}, \infty).$$

It is known that $\Gamma_0(2)$ is free on two generators and one can take a random finite index n subgroup Γ of $\Gamma_0(2)$. X_Γ is an n-cover of the surface $X_{\Gamma_0(2)}$.

Conjecture 2.1. *Alon's Conjecture:* $\forall \epsilon > 0$ *a random* n*-cover, (i.e. a quotient by a finite subgroup of index* n*)* X_Γ *satisfy*

$$Spec_\Delta(\mathbb{H}) \subseteq [\frac{1}{4} - \epsilon, \infty)$$

Namely it is a $\frac{1}{4} - \epsilon$ *expander, with probability* $1 - o(1)$.

Conjecture 2.2. *Bilu-Linial Conjecture: There is a Ramanujan n-cover for every n.*

Remark 2.3. *Selberg's $\frac{1}{4}$ Conjecture: For a congruence subgroup $\Gamma_0(N)$, $X_{\Gamma_0(N)}$ is Ramanujan.*

3. Irit Dinur

3.1. **Agreement Expansion.** Let $[n]$ be a set of vertices and $X \subseteq \mathrm{P}([n])$ a collection of subsets. Take for each $S \in X$ a (local) function $f_S \colon S \to \mathbb{F}_2$. We say that f_{S_1}, f_{S_2} agree locally if $f_{S_1}|_{S_1 \cap S_2} = f_{S_2}|_{S_1 \cap S_2}$. If all the pairs of local functions f_S agree locally, then there is a global function $F \colon [n] \to \mathbb{F}_2$ where $F|_S = f_S$. In this case we say that the ensemble of functions $\{f_S\}_{S \in X}$ is global.

The choice of range $\mathbb{F}_2$ for the functions is arbitrary and can be replaced by any other finite (or infinite?) set.

We are interested in a quantitative relation between the amount of local agreement and closeness to being global. Concretely, for a distribution D on pairs $(S_1, S_2) \in X^2$, we define the amount of local agreement as

$$agree_D(\{f_S\}) = Pr_{S_1,S_2 \sim D}[f_{S_1}|_{S_1 \cap S_2} = f_{S_2}|_{S_1 \cap S_2}]$$

The distance from being global is

$$dist(\{f_S\}, global) = \min_{G:[n] \to \mathbb{F}_2} Pr_S[f_S \neq G|_S]$$

It is known [IDSS] that for the complete k-uniform hypergraph on n vertices, i.e. $X = \binom{[n]}{k}$, there is a distribution D for which

$$agree_D(\{f_S\}) > 1 - \epsilon \quad \Rightarrow \quad distance(\{f_S\}, global) < O(\epsilon)$$

The same was also recently proven for a family of bounded-degree high dimensional complexes [IDTK], using the fact that the links are sufficiently good expanders. In [IDTK] a structure X for which this holds is called an *agreement expander*. We ask here about a quantitatively stronger agreement expansion property, namely

Question 3.1 (Strong Agreement Expander)**.** *What can be said when $agree(\{f_S\}) > \delta$, or even $agree(\{f_S\}) \geq 0.5$? Does it imply that $\{f_S\}$ is correlated with a global ensemble, i.e. $distance(\{f_S\}, global) < 1 - \delta'$ for some constant $\delta' > 0$?*

More broadly, is there any bounded-degree structure X in which such a theorem holds?

When $agree(\{f_S\}) \geq 0.5$ we can no longer expect a single global function to agree with $\{f_S\}$ as can be seen by taking two functions $G_1, G_2 : [n] \to \mathbb{F}_2$ and for each S deciding independently at random whether to assign $f_S := G_1|_S$ or $f_S := G_2|_S$. The ensemble $\{f_S\}$ has $agree(\{f_S\}) \approx 1/2$ and its correlation with the closest global function is about $1/2$ as well.

For the complete hypergraph such a theorem is known to be true, but is wide open for X with $|X| = O(n)$.

4. Leonard Schulman

4.1. Non-Palindrome Coloring of Regular Trees.

Question 4.1. *Let T_3 be the 3-regular tree. Let $w = w_1...w_n, \quad w_i \in [k]$ be a word. Define* $\mathrm{DPal}(w) = \frac{d_{\mathrm{Hamming}}(w,\overline{w})}{n}$*, where $\overline{w} = w_n...w_1$. There exists $\epsilon > 0$, natural number k and a function $\chi\colon \mathrm{Edges}(T_3) \to [k]$ such that for all words w of length greater than 1 that you read on a simple path in T_3 we have $\mathrm{DPal}(w) > \epsilon$. Problem: give an explicit construction. (Specifically, fixing a root, you should be able to compute the label of an edge at distance n from the root in time poly(n).)*

Notice that if $\epsilon = 0$ it is enough to avoid palindromes of length 2 and 3 and this is easy.

5. Noam Lifshitz

5.1. Tensor Powers of Simplicial Complexes. Given a graph $G = (V, E)$ its tensor power, $G^{\otimes n}$, is the graph with vertex set V^n and edges between two n-tuples $(v_1, ..., v_n), (u_1, ..., u_n)$ if $(v_i, u_i) \in E$ for all $1 \leq i \leq n$. We can define a tensor power of a (weighted) simplical complex in a similar way.

Question 5.1. *For which simplicial complexes X can we determine the largest independent set in $X^{\otimes n}$?*

Remark 5.1. *The Turán problem is a special case of this question: take the complex with vertices $\binom{[n]}{k}$ and the simplices of highest dimension corresponding to copies of a given hyper-graph. It turns out that this complex is a tensor power of a given complex.*

There are two alternative notions for a set A of vertices to be almost independent.

(1) You can delete a "small portion" of the vertices of A to turn it into an independent set.

(2) A contains "few" simplices of the highest dimension.

Conjecture 5.2. *Fix a simplicial complex X. For each $\epsilon > 0$ there exists some $\delta > 0$ such that the following holds: Let A be some set of vertices. Suppose that a simplex of the highest dimesnion σ is contained in A with probability at most δ, then there exists an independent set B, such that $\|A \Delta B\| < \epsilon$.*

Remark 5.3. *Conjecture 5.2 implies the hypergraph removal lemma.*

Question 5.2. *What conditions on X imply that $X^{\otimes n}$ is a co-boundary expander? a co-systolic expander? etc.*

6. Uriya First

6.1. **Independence of Different Ramanujan Properties.** Let G be a simple algebraic group over a non-Archimedian local field, e.g. $\mathrm{PGL}_d(F)$. Let X be the corresponding affine Btuhat-Tits building. Suppose X has finite quotients.

Question 6.1. *Is there a cocompact lattice $\Gamma \leq G$ and $0 \leq i, j \leq d$ such that $\Gamma \setminus X$ is Ramanujan in dimension i and not in dimension j in the sense of* [Fir16].

Question 6.2. *Is there a cocompact lattice $\Gamma \leq G$ and $0 \leq i, j \leq d$ such that $\Gamma \setminus X$ is Ramanujan relative to the i-th dimensional Laplacian and not relative to the j-th dimensional Laplacian.*

This question can be posed for other operators associated with the complex, although allowing arbitrary combinatorial operators eventually leads to examples that are Ramanujan relative to one operator and not relative to another.

Question 6.1 can be rephrased in representation-theoretic language: Let $x_1, \ldots, x_n$ be representatives for the G-orbits in $X^{(i)}$ and let $y_1, \ldots, y_m$ be representatives for the G-orbits in $X^{(j)}$. Write $K_t = \mathrm{Stab}_G(x_t)$ and $L_s = \mathrm{Stab}_G(y_s)$; these are compact open subgroups of G. Is there a cocompact lattice $\Gamma \leq G$ such that every irreducible infinite dimensional subrepresentation $V \leq \mathrm{L}^2(\Gamma \setminus G)$ with $V^{K_1} + \cdots + V^{K_n} \neq 0$ is tempered, but there is a non-tempered irreducible infinite dimensional $V \leq \mathrm{L}^2(\Gamma \setminus G)$ such that $V^{L_1} + \cdots + V^{L_m} \neq 0$.

The Ramanujan complexes of Lubotzky-Samuels-Vishne, Li, First are completely Ramanujan and hence cannot serve as counterexamples. However, perhaps some of the non-Ramanujan complexes in LSV can serve as candidates.

7. Gábor Damásdi

7.1. **Polytope Equi-partition.**

Theorem 7.1. *(Boros - Füredi, Bárány) If you take n points in the plane there exists a point p that is covered by at least $\frac{2}{9} \cdot \binom{n}{3}$ of the triangles defined by the points.*

There is a similar result for higher dimensional point sets.

Boris Bukh proves this Theorem using the following Lemma:

Lemma 7.2. *For any set of n points in the plane there exists 3 concurrent lines that separate the point set into 6 equal parts.*

Question 7.1. *Is there a higher dimensional analogue of this Lemma, namely a point and a polytope such that the cones defined by the facets of the polytope and the point separate the point set into equally sized subsets.*

In the 2-dim case there exists an affine image of the regular hexagon together with its center that gives the lines of Bukh's Lemma.

Conjecture 7.3. *(Makeev, 2004) Let $\Delta_d \subseteq \mathbb{R}^d$ be a regular simplex. Let*

$$T_d = \Delta_d - \Delta_d = \{x - y \mid x, y \in \Delta_d\}.$$

Then there exists an affine image of T_d together with its center point that divides our set point into equal sized subsets.

8. Shai Evra

8.1. **High Dimensional Expanders.**

(1) Is there a random model generating bounded degree 2-dim complexes which give co-boundary expanders with high probability?
(2) Is there an explicit construction of bounded degree co-boundary expanders? (Can the degree be 7 for example?)
(3) Is there a 2-dim bounded degree complex, $X = (V, E, T)$, which satisfy the following discrepancy property: For any $A_1, A_2, A_3 \subset V$, then
$$\left| |T(A_1, A_2, A_3)| - \frac{k \cdot |A_1||A_2||A_3|}{|V|} \right| \leq \lambda \cdot \sqrt{|A_1||A_2||A_3|}$$
Where λ/k tends to zero where V grows to infinity, and k is the degree.
(4) Let X be a 2-dim bounded degree complex and let G be the induced graph whose vertexes are the edges of X and the edges are triangles of X. Can you prove a useful mixing lemma for this graph G?

where useful means that you can derive from it good discrepancy or coboundary properties for say a Ramanujan complex.

(5) Let X be a pure connected complex, all of whose proper links have vanishing cohomology and good spectral expansion. Does X posses the topological overlapping property?

(6) Is it true that every d-dim coboundary expander does not embed into R^{2d}?

(7) Let $k < d$. Find the optimal $c_{k,d}$ such that for any continuous mapping of the d-dim complete complex into R^{d+k}, there is a k-dim hyperplane whose preimage is of size $c_{k,d} \cdot \binom{n}{d+1}$.

(8) Does Z-coboundary expander imply F_p-coboundary expander (using Hamming norms)?

References

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[IDSS] I. Dinur and S. Steurer, *Direct Product Testing*, https://eccc.weizmann.ac.il/report/2013/179/, 188–196, DOI 10.1109/CCC.2014.27.

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