Lecture 4 - Andreas Thom

C^*-algebras and groups

Def: A C^*-algebra is a norm closed involutive subalgebra of $\mathcal{B}(\ell^2(\mathbb{N})) = \{T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ bounded linear op.$\}$

T bdd. $\iff \exists\, C > 0$ s.t. $\|T\xi\| \le C\|\xi\|$

$\xi \in \ell^2(\mathbb{N}) \qquad \|\xi\| = \left(\sum_{i=1}^{\infty} |\xi_i|^2\right)^{1/2}$

$\|T\|_{op} = \sup_{\|\xi\| \le 1} \|T\xi\|$

$\|TS\|_{op} \le \|T\|_{op}\|S\|_{op}$

$\|T+S\|_{op} \le \|T\|_{op} + \|S\|_{op}$

If T is a bdd operator $\exists!\, T^* \in \mathcal{B}(\ell^2(\mathbb{N}))$ s.t.

$\langle T\xi, \eta\rangle = \langle \xi, T^*\eta\rangle \quad \forall \xi, \eta \in \ell^2(\mathbb{N})$

$*$ is an involution

$(T^*)^* = T$

$(TS)^* = S^*T^*$

$(\lambda T)^* = \bar{\lambda} T^*$

Examples: 1. X compact top. space, μ measure on X

$$L^2(X,\mu) \qquad f \in C(X) \subseteq \mathcal{B}(L^2(X,\mu))$$

$$f \longmapsto M_f : L^2 \to L^2$$
$$\xi \longmapsto f \cdot \xi$$

$$\|M_f\|_{op} = \|f\|_{\infty,\mu}$$

2. $L^\infty(X,\mu) \subseteq \mathcal{B}(L^2(X,\mu))$

orthonormal $\overline{\operatorname{span}\{\delta_h : h \in \Gamma\}}$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \shortparallel$

3. $\underline{\Gamma \text{ group}}$: Consider $\ell^2\Gamma$. For every $g \in \Gamma$ define $\lambda(g) : \ell^2\Gamma \to \ell^2\Gamma$

$$\lambda(g)\,\delta_h = \delta_{gh}$$

$$\|\lambda(g)\| = 1$$

$$\lambda(g)^* = \lambda(g)^{-1} = \lambda(g^{-1}) \qquad \text{unitary}$$

$$\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma) = \{u \in \mathcal{B}(\ell^2\Gamma) : u^*u = 1 = uu^*\}$$

$$C^*_{red}(\Gamma) := \overline{\operatorname{Span}}^{\|\cdot\|_{op}}\{\lambda(g) : g \in \Gamma\} \quad \text{reduced } C^*\text{-algebra.}$$

In general: $\pi : \Gamma \to \mathcal{U}(\ell^2(\Gamma))$ is a unitary rep if

$$\pi(gh) = \pi(g)\pi(h) \quad \text{and} \quad \pi(g)^* = \pi(g)^{-1} \quad \forall g,h \in \Gamma$$

$$C^*(\Gamma,\pi) = \overline{\operatorname{Span}}^{\|\cdot\|_{op}}\{\pi(g) : g \in \Gamma\}$$

Let π: be the direct sum of all unitary reps. and set

$C^*(\Gamma) = C^*(\Gamma, \pi)$, maximal group C^*-algebra.

Prop: $\forall \pi : \Gamma \to \mathcal{U}(\ell^2(\mathbb{N}))$ $\exists !$ $\varphi_\pi : C^*(\Gamma) \twoheadrightarrow C^*(\Gamma, \pi)$

Thm: TFAE

1) The trivial rep. extends to $C^*_{red}(\Gamma)$, i.e. $\exists\, \varphi_1 : C^*_{red}(\Gamma) \to \mathbb{C}$

2) $\varphi_\lambda : C^*(\Gamma) \xrightarrow{\sim} C^*_{red}(\Gamma)$

3) Γ is amenable.

Examples:

1) $\Gamma = \mathbb{Z}$ $\quad C^*_{red}(\mathbb{Z}) = C^*(\mathbb{Z}) = C(S^1)$ Pontryagin dual

$$\lambda_n \longmapsto M_{e^{2\pi i n t}}$$

2) $\mathbb{F}_2$ $\quad C^*_{red}(\mathbb{F}_2)$ simple with unique trace $\quad (\tau(ab) = \tau(ba))$

$$\tau : C^*_{red}(\mathbb{F}_2) \longrightarrow \mathbb{C} \qquad g \longmapsto \begin{cases} 1 & g = 1 \\ 0 & \text{otherwise} \end{cases}$$

~~Prop: $C^*(\mathbb{F}_2)$ is~~

Def: A group Γ is called residually finite dimensional (RFD) $\iff$

$$\|a\|_{univ} = \sup_{\substack{\pi \text{ finite} \\ \text{dim unitary} \\ \text{rep.}}} \|\pi(a)\| \qquad \forall a \in \mathbb{C}\Gamma$$

Examples:

(1) $\mathbb{F}_n$ are (RFD)

(2) Surface groups are (RFD)

(3) $SL_3(\mathbb{Z})$ is not (RFD)

Open problem: Is $\mathbb{F}_2 \times \mathbb{F}_2$ (RFD) ?

Observation: If Γ is (RFD) $\Rightarrow$ Given $a \in \mathbb{Z}\Gamma$ $\exists$ computable seq. of increasing lower bounds for $\|a\|_{univ}$ converging to it.

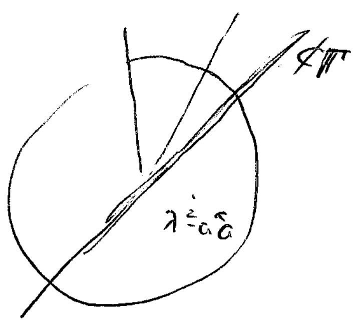
$$\|a\|^2_{univ} = \|a^*a\|_{univ} \leq \lambda^2 \iff \lambda^2 - a^*a \geq 0$$

$$\iff \lambda^2 - a^*a + \varepsilon = \sum_{i=1}^{l} a_i^* a_i \quad \text{for } a_i \in \mathbb{C}\Gamma. \quad \forall \varepsilon > 0$$

Suppose that $\lambda^2 - a^*a$ is not a sum of squares Σ^2

$\exists\, \varphi : \mathbb{C}\Gamma \longrightarrow \mathbb{C}$ s.t. $\varphi(b^*b) \geq 0$ $\forall b \in \mathbb{C}\Gamma$

and $\varphi(\lambda^2 - a^*a) < 0 \Rightarrow$ $\varphi|_\Gamma : \Gamma \to \mathbb{C}$ is a positive definite function.



$\rightsquigarrow$ GNS construction

$\langle b, b' \rangle = \varphi(b^* b') \rightsquigarrow \mathcal{H}_\varphi$ is a Hilbert space + unit rep.

<u>Defn:</u> Let Γ be f.g. group with generator set $S = S^{-1}$

$$M_S = \frac{1}{|S|} \sum_{s \in S} s \quad \in \quad \mathbb{C}\Gamma$$

<u>Prop:</u> Γ has $(T) \Longleftrightarrow \exists \varepsilon > 0$ s.t. $Sp(\pi(M_S)) \subseteq [-1, 1-\varepsilon] \cup \{1\}$

$\forall$ unitary rep. $\Longleftrightarrow \Delta_S^2 - \varepsilon \Delta_S \geq 0$ in $C^*(\Gamma)$

$(\Delta_S = 1 - M_S)$ $\underset{\text{Ozawa}}{\Longleftrightarrow} \Delta_S^2 - \varepsilon \Delta_S = \sum_{i=1}^{l} a_i^* a_i$ in $\mathbb{C}\Gamma$

$\hookrightarrow$ you don't need $+\varepsilon' > 0$ but $+\varepsilon' \Delta_S$ is enough.

<u>example:</u> $SL(3, \mathbb{Z})$ $\quad \varepsilon \geq \frac{1}{12}$

Group stability and HD expanders (1)

Andreas Thom - Lecture 5

C^* algebras and groups

$C(X) \subseteq L^\infty(X,\mu) \subseteq \mathcal{B}(L^2(X,\mu))$

Γ group $\quad \lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma) \qquad \lambda(h)\delta_g = \delta_{hg}$

$\ell^2\Gamma \to \overline{\mathrm{span}\{\delta_g : g\in\Gamma\}}$

$C^*_{red}(\Gamma) = \overline{\mathrm{span}\{\lambda(g) : g\in\Gamma\}}^{\|\cdot\|_{op}}$

$\cup\!\!\mid$

$\mathbb{C}\Gamma$

$\tau(a) = \langle a\delta_e, \delta_e\rangle \in \mathbb{C} \qquad$ "trace"

$\tau : C^*_{red}(\Gamma) \longrightarrow \mathbb{C}$

(1) $a\in\mathbb{C}\Gamma \qquad a = \sum_g a_g g \implies \tau(a) = a_e$

(2) $\tau(ab) = \tau(ba) \qquad \forall a,b\in C^*_{red}(\Gamma)$

(3) $a \geq 0$, $\tau(a) = 0 \implies a = 0$.

(4) $\tau(1) = 1$

(5) τ is linear

Proof of (3): $a\geq 0 \iff a = b^*b$ for some $b\in C^*_{red}(\Gamma)$

$0 = \tau(a) = \tau(b^*b) = \langle b^*b\delta_e, \delta_e\rangle = \langle b\delta_e, b\delta_e\rangle = \|b\delta_e\|^2$

$\implies b\delta_e = 0$. The operator commute with right multiplication $\rho(g)b = b\rho(g)\ \forall g\in\Gamma$

$\implies b\delta_g = 0 \ \forall g\in\Gamma \implies b = 0.$

$\exists$ faithful trace as above $\Longrightarrow \forall n\in\mathbb{N}, \forall a,b\in M_n(C^*_{red}(\Gamma)) \subseteq B(l^2\Gamma^{\oplus n})$

$a\cdot b = 1\!\!1_n \implies b\cdot a = 1\!\!1_n$

<u>Proof:</u> $a\cdot b = 1\!\!1_n \implies baba = ba$, i.e. ba is idempotent. Suppose in addition $(ba)^* = ba$ (this can be checked), then $1-ba \geq 0$ is idempotent and self adjoint, but $\tau(1-ba) = 1-\tau(ba) = 1-\tau(ab) = 1-\tau(1) = 0$

$\implies 1 = ba$ ∎

The last property has a name: $C^*_{red}(\Gamma)$ is stably finite.

<u>Defn:</u> A C^*-algebra A is called MF (matricially finite) if $\forall a_1,\dots,a_n \in A$ and $p_1,\dots,p_m \in \mathbb{C}\langle x_1,\dots,x_n\rangle$ (non-commutative polynomials)

$\forall \varepsilon>0 \ \exists A_1,\dots,A_n \in M_k\mathbb{C}$ s.t.

$\forall 1\leq i\leq m \quad \big|\, \|P_i(a_1,\dots,a_n)\|_A - \|P_i(A_1,\dots,A_n)\|_{M_k\mathbb{C}} \big| < \varepsilon.$

<u>Claim (without proof):</u> MF C^*-algebra is stably finite.

<u>Conj (Kirchberg):</u> a stably finite C^*-algebra is MF.

<u>Prop:</u> If $C^*_{red}(\Gamma)$ is MF, then $\forall F\subseteq\Gamma$ finite and $\varepsilon>0$ $\exists K$ finite dim and $\exists \varphi: F \to \mathcal{U}(K)$ (not homomorphism) which is almost multiplicative

$\|\varphi(gh) - \varphi(g)\varphi(h)\| < \varepsilon \quad \forall g,h\in F \qquad \|\varphi(g) - 1\!\!1\| \geq \frac{1}{2} \quad \forall g\in F\setminus\{1\}.$

<u>Thm</u>: The conj is true for $\Gamma = \mathbb{F}_2$ and for Γ amenable.

Von Neumann algebras

Observe $\mathcal{B}(\ell^2\mathbb{N})$. There are several ~~metrics~~ topologies one can give it

$x_n \to x$ in op $\iff$ $\|x_n - x\|_{op} \to 0$

$x_n \to x$ in SOT $\iff$ $\forall \xi \in \ell^2\mathbb{N}$ $\|x_n\xi - x\xi\|_{\ell^2(\mathbb{N})} \to 0$

$x_n \to x$ in WOT $\iff$ $\forall \xi, \eta \in \ell^2\mathbb{N}$ $\langle x_n\xi, \eta\rangle \to \langle x\xi, \eta\rangle$.

Op $\to$ SOT $\to$ WOT

<u>Defn</u>: $A \subseteq \mathcal{B}(\ell^2\mathbb{N})$ is called a Von-Neumann algebra if it is a $*$-algebra with unit (the unit of $\mathcal{B}(\ell^2\mathbb{N})$) which is closed in the SOT top.

<u>Rk</u>: vN algebra $\Rightarrow$ C^*-algebra

<u>Thm (Von Neumann)</u>: Let $A \subseteq \mathcal{B}(\ell^2\mathbb{N})$ be a $*$-algebra with unit. TFAE

1) A is SOT-closed
2) A is WOT-closed
3) $A = A''$, where for $S \subseteq \mathcal{B}(\ell^2\mathbb{N})$ $S' = \{T \in \mathcal{B}(\ell^2\mathbb{N}) : sT = Ts \ \forall s \in S\}$

<u>Rk</u>: Trivially $A \subseteq A'' = (A'')''$ $\Rightarrow$ A'' is a von-Neumann algebra. The smallest on containing A.

Defn: $L\Gamma := \overline{\mathrm{Span}\{\lambda(g) : g \in \Gamma\}}'' = \lambda(\Gamma)'' \supseteq C^*_{red}(\Gamma)''$

One can still define $\tau : L\Gamma \to \mathbb{C}$ $\tau(a) = \langle a\delta_e, \delta_e\rangle$ which is a faithful state.

Rem: L^2-Betti numbers.

$$\beta_k^{(2)}(\Gamma) = \dim_{L\Gamma} H^k(\Gamma; \ell^2\Gamma)$$

$$\|a\|_2 := \tau(a^*a)^{1/2} = \|a\delta_e\|_{\ell^2 \mathbb{N}}$$

Prop: Sequences satisfying $\|\cdot\|_{op}$-bnd, $\|\cdot\|_2$-norm conv. have a limit in $L\Gamma$.

$\Rightarrow (\mathcal{U}(L\Gamma), \|\cdot\|_2)$ is a Polish group.

Another example: $(M_n\mathbb{C}, \frac{1}{n}\mathrm{tr})$

Defn: (M, τ) a tracial von-Neumann algebra is called Connes approximated $\Leftrightarrow \forall a_1, \dots a_n \in M$ $\forall f_1, \dots f_m \in \mathbb{C}\langle x_1, \dots x_n\rangle$ $\forall \varepsilon > 0$ $\exists A_1, \dots, A_n \in M_k\mathbb{C}$ s.t.

$$\forall\, 1 \leq i \leq m \quad \left| \tau(f_i(a_1, \dots a_n)) - \tfrac{1}{k}\mathrm{tr}(f_i(A_1, \dots A_n)) \right| < \varepsilon$$

Prop: $L\Gamma$ Connes approx. $\Longrightarrow$ $\forall F \subseteq \Gamma$ finite $\forall \varepsilon > 0$

$\exists \varphi : \Gamma \longrightarrow \mathcal{U}(k)$ $\quad \|\varphi(gh) - \varphi(g)\varphi(h)\|_2 < \varepsilon \quad \|\varphi(g) - id\|_2 \geq \frac{1}{2} \;\; \forall g \neq id \in F$

$\forall g, h \in F$

$\Longleftrightarrow \Gamma$ is hyperlinear.

Conj (Connes): Γ has (T) $\qquad L\Gamma \cong L\Lambda \Longrightarrow \Gamma \cong \Lambda$

Evidence: Γ residually finite

Γ amenable $\quad$ (Thm: Γ, Λ ∞-conjugacy classes amenable
$(L\Gamma, \tau) \cong (L\Lambda, \tau)$
hyperfinite II_1-factor)

Conj: $n \neq m \Longrightarrow L\mathbb{F}_n \ncong L\mathbb{F}_m$

Conj: $L\Gamma \cong L\Lambda \Longrightarrow \beta_1^{(2)}(\Gamma) = \beta_1^{(2)}(\Lambda)$.

Lecture 6

Cohomology of simplicial complexes

Let V be a (usually finite) set and $X \subseteq \mathcal{P}(V)$
X is called a simplicial complex if it is closed under inclusion, i.e. $F \in X$, $G \subseteq F \Rightarrow G \in X$.

WLOG we assume that $V \subseteq X$, i.e., $v \in X$ $\forall v \in V$.

- $F \in X$, $\dim(F) = |F| - 1$.
- elements of X are called faces.
- $d = \dim(X) = \max\{\dim F : F \in X\}$
- X is called pure if $\forall F \in X$ $\exists G \in X$ s.t. $\dim G = \dim X$ and $F \subseteq G$.

Recall: An hypergraph on V is any subset of $\mathcal{P}(V)$
a k-uniform hypergraph is an hypergraph all of its subsets are in $\binom{V}{k}$.

Y k-uniform hypergraph $\longrightarrow X(Y) = \{G \subseteq V : G \subseteq A \text{ for some } A \in Y\}$
is a s.c.

X a pure s.c. $\quad Y = Y(X) = \{G \in X : \dim G = \dim X\}$
is a $\dim X$-uniform hypergraph.

$F = \{v_{i_0}, \dots, v_{i_k}\}$ a face of dim k in X.

An ordered face $\overline{F} = [v_{i_0}, \dots, v_{i_k}]$

$X(k) = \{F \subset X : \dim F = k\}$

$\vec{X}(k) = \{\text{ordered faces of dim } k\}$

$|\vec{X}(k)| = (k+1)!\ |X(k)|$

Let M be an Abelian group. Denote

$$C^k(X;M) = \left\{ f: \vec{X}(k) \to M \ :\ f([v_{\sigma(i_0)}, \dots v_{\sigma(i_k)}]) = \operatorname{sgn}(\sigma) f([v_{i_0}, \dots, v_{i_k}]) \ \ \forall \sigma \in S_{(k+1)} \right\}$$

the space of antisymmetric functions.

<u>Note that:</u> $C^k(X;M) \simeq M^{|X(k)|}$ as algebras

If $M = \mathbb{F}$ is a field $\dim_{\mathbb{F}} C^k = |X(k)|$

<u>The cohomology map</u>

$\delta_k = d_k : C^k \longrightarrow C^{k+1}$

$$\delta_k \varphi([v_{i_0}, \dots, v_{i_{k+1}}]) = \sum_{j=0}^{k+1} (-1)^j \varphi([v_{i_0}, \dots v_{i_{j-1}}, v_{i_{j+1}}, \dots v_{i_{k+1}}]) \qquad \forall \varphi \in C^k$$

v_{i_j} ommited

<u>Ex:</u> $\delta_k \varphi \in C^{k+1}$ $\forall \varphi \in C^k$.

Ex: $\delta_k \circ \delta_{k-1} = 0$

Cor: $\operatorname{Im}\delta_{k-1} =: \underbrace{B^k(X;M)}_{k\text{-coboundaries}} \subset \underbrace{Z^k(X;M)}_{k\text{-cocycles}} = \ker \delta_k$

$H^k(X;M) = Z^k(X;M) / B^k(X;M)$ the k-cohomology group of X over M.

E. If F is a field $\sum_{i=0}^{d} (-1)^i \dim H^i(X;F) = \sum_{i=-1}^{d} (-1)^i f(i)$,

where $f(i) = |X(i)|$. This is the Euler characteristic of X, denoted $\chi(X)$.

Hint: $C^{-1} \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \dots \longrightarrow C^d$

calculate dim using the fact that $|X(i)| = \dim_F C^i(X;F)$.

Example: $X =$ a graph $M = \mathbb{F}_2$

$X(-1) = \{\emptyset\}$

$X(0) = V$ $|V| = n$

If X is k-regular $|X(1)| = \frac{kn}{2}$

$B^0 = \operatorname{Im}\delta_{-1} = \{\vec{0}, \vec{1}\}$ constant functions $C^0 = \{f : V \to \mathbb{F}_2\} = \{1\!\!1_y : y \in V\}$

$Z^0 = \ker \delta_0 = \{f : V \to \mathbb{F}_2 : \delta f = 0\} = \{$functions constant on connected comp.$\}$.

$\dim Z^0 = $ # conn. comp.

$\dim H^0 = $ # conn. comp $- 1$

$H^0 = \{0\} \iff X$ is connected.

Assume now X is connected.

$$-|X(1)| + |X(0)| - |X(-1)| = \dim H^0 - \dim H^1$$

$$-1 + n - \frac{kn}{2} = 0 - \dim H^1$$

$$\boxed{\dim H^1 = \frac{kn}{2} - (n+1)}$$ the num of edges outside a spanning tree

<u>Defn:</u> Coboundary expansion

$F \in X(k)$ $\qquad d = \dim X$

$$w(F) = \frac{\deg F}{\binom{d+1}{k+1}|X(d)|}, \text{ where } \deg(F) = |\{G \in X(d) : F \subseteq G\}|.$$

Note that $\sum_{F \in X(k)} w(F) = 1$, i.e. $w(F)$ is a prob measure on $X(k)$.

$F = \mathbb{F}_2$

Define: k-th expansion of X to be

$$\tilde{h}_k(X) = \min_{f \in C^k \setminus B^k} \frac{\|\delta_k f\|}{\|[f]\|}, \quad \text{where } \|f\| = |\mathrm{Supp} f| \quad \forall f \in C^k$$

$$|[f]| = \min_{b \in C^{k-1}} |f + \delta b|$$

$$h_k(X) = \min_{f \in C^k \setminus B^k} \frac{\|\delta_k f\|}{\|[f]\|}, \quad \text{where } \|f\| = \sum_{F \subset \mathrm{Supp}(f)} w(F).$$

Prop: TFAE (X finite)

(1) $h_k(X) > 0$

(2) $\tilde{h}_k(X) > 0$

(3) $H^k(X; \mathbb{F}_2) = 0$

Example: X a k-regular graph

$$\tilde{h}_0(X) = \min_{f \in C^0 \setminus B^0} \frac{|\mathrm{Supp}\, \delta f|}{|[f]|} = \min_{\substack{A \subset V \\ A \neq \emptyset, V}} \frac{|E(A, A^c)|}{\min\{|A|, |A^c|\}}$$

Thm: (L-M, Gromov) X d-dim complete s.c. on n vertices, then

$$h_k(X) \geq 1 \text{ for } 0 \leq k \leq d-1$$